

# Teleportation capability, distillability, and nonlocality on three-qubit states

Soojoon Lee,<sup>1,\*</sup> Jaewoo Joo,<sup>2,†</sup> and Jaewan Kim<sup>3,‡</sup>

<sup>1</sup> Department of Mathematics and Research Institute for Basic Sciences, Kyung Hee University, Seoul 130-701, Korea

<sup>2</sup> Blackett Laboratory, Imperial College London, Prince Consort Road, London, SW7 2BW, United Kingdom

<sup>3</sup> School of Computational Sciences, Korea Institute for Advanced Study, Seoul 130-722, Korea

(Dated: February 28, 2007)

In this work, we consider teleportation capability, distillability, and nonlocality on three-qubit states. In order to investigate two relations among them, we introduce the class of 3-qubit states with 4 parameters, presented by Dür *et al.* [Phys. Rev. Lett. **83**, 3562 (1999)], into which any of 3-qubit states can be transformed by local quantum operation and classical communication. On the class, we explicitly compute the quantities about the maximal teleportation fidelity, the distillability of the Greenberger-Horne-Zeilinger state, and the maximal violation of the Mermin inequality, and finally show that if any 4-parameter 3-qubit state in the class violates the Mermin inequality then the state is useful for 3-qubit teleportation, and that if any 3-qubit state in the class is useful for 3-qubit teleportation then the state can distill the Greenberger-Horne-Zeilinger state.

PACS numbers: 03.67.-a, 03.65.Ud, 03.67.Mn 03.67.Hk,

Teleportation capability, distillability, and nonlocality have been considered as significant features of quantum entanglement, and have been helpful to understand quantum entanglement. The three features have been known as a practical application of quantum entanglement, an important method to classify quantum entanglement with respect to the usefulness for quantum communication, and a physical property to explain the quantum correlation, respectively.

In the case of 2-qubit states, it has been shown that there are two relations among the three features: If any 2-qubit state is useful for teleportation then it is distillable into a pure entanglement, and if any 2-qubit state violates the Bell inequality then it is useful for teleportation [1]. Thus one could naturally ask what relations exist for multiqubit states. In this work, we investigate teleportation, distillability, and nonlocality on 3-qubit states in order to answer the question, since the 3-qubit case could be readily generalized into the multiqubit case. On this account, we appropriately define the concepts of teleportation, distillability, and nonlocality on 3-qubit states, meaningfully present the quantities obtained from them, and compare the quantities to look into the relations.

For teleportation over 3-qubit states, we recall the Hillery-Bužek-Berthiaume (HBB) [2] protocol, which is the splitting and reconstruction of quantum information over the Greenberger-Horne-Zeilinger (GHZ) state [3] by local quantum operations and classical communication (LOCC). The protocol can be modified into a teleportation protocol over a general 3-qubit state in the compound system 123, as presented in [4]. The modified protocol is illustrated in Fig. 1 and is described as follows: Let  $i$ ,  $j$ , and  $k$  be distinct in  $\{1, 2, 3\}$ . (i) Make

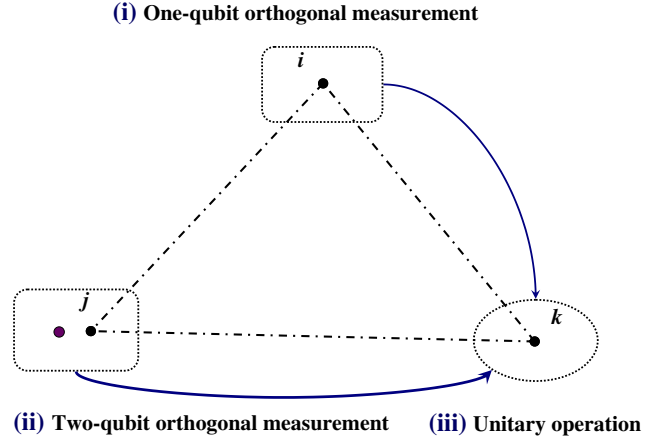


FIG. 1: The modified teleportation protocol over a 3-qubit state presented in [4]: The dotted boxes and ellipse represent performing the orthogonal measurements and applying the unitary operation, respectively. The arrows represent sending classical information corresponding to the measurement results.

a one-qubit orthogonal measurement on the system  $i$ . (ii) Prepare an arbitrary one-qubit state, and then make a two-qubit orthogonal measurement on the one qubit and the system  $j$ . (iii) On the system  $k$ , apply a proper unitary operation depending on the 3-bit classical information of the two above measurement outcomes. We remark that the modified protocol is essentially equivalent to the original one with respect to the splitting and the reconstruction of quantum information.

As mentioned in [4], it is noted that any observable for a one-qubit measurement can be described as

$$U^\dagger \sigma_3 U = U^\dagger |0\rangle\langle 0| U - U^\dagger |1\rangle\langle 1| U, \quad (1)$$

where  $\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$  is one of Pauli matrices, and  $U$  is a  $2 \times 2$  unitary matrix. Thus, after the step (i)

\*Electronic address: level@khu.ac.kr

†Electronic address: jaewoo.joo@imperial.ac.uk

‡Electronic address: jaewan@kias.re.kr

of the teleportation protocol over a given 3-qubit state  $\rho_{123}$ , the resulting 2-qubit state of the compound system  $jk$  becomes

$$\begin{aligned} \varrho_{jk}^t &\equiv \frac{\text{tr}_i \left( U_i^\dagger |t\rangle \langle t| U_i \otimes I_{jk} \rho_{123} U_i^\dagger |t\rangle \langle t| U_i \otimes I_{jk} \right)}{\langle t| U_i \rho_i U_i^\dagger |t\rangle} \\ &= \frac{\text{tr}_i \left( |t\rangle \langle t| U_i \otimes I_{jk} \rho_{123} U_i^\dagger |t\rangle \langle t| \otimes I_{jk} \right)}{\langle t| U_i \rho_i U_i^\dagger |t\rangle} \end{aligned} \quad (2)$$

with probability  $\langle t| U_i \rho_i U_i^\dagger |t\rangle$  for each  $t = 0, 1$ , where  $U_i$  is a  $2 \times 2$  unitary matrix of the system  $i$ , and  $\rho_i = \text{tr}_{jk}(\rho_{123})$ .

we now review the properties of the teleportation fidelity [5], which represents the faithfulness of a teleportation over a 2-qubit state, and the fully entangled fraction [1, 6, 7, 8]. The teleportation fidelity is naturally defined as

$$F(\Lambda_\rho) = \int d\xi \langle \xi | \Lambda_\rho(|\xi\rangle \langle \xi|) | \xi \rangle, \quad (3)$$

where  $\Lambda_\rho$  is a given teleportation protocol over a 2-qubit state  $\rho$ , and the integral is performed with respect to the uniform distribution  $d\xi$  over all one-qubit pure states, and the fully entangled fraction of  $\rho$  is defined as

$$f(\rho) = \max \langle e | \rho | e \rangle, \quad (4)$$

where the maximum is over all maximally entangled states  $|e\rangle$  of 2 qubits. It has been shown [7, 8] that the maximal fidelity achievable from a given bipartite state  $\rho$  is

$$F(\Lambda_\rho) = \frac{2f(\rho) + 1}{3}, \quad (5)$$

where  $\Lambda_\rho$  is the standard teleportation protocol over  $\rho$  to attain the maximal fidelity. We remark that  $F(\Lambda_\rho) > 2/3$  (or  $f(\rho) > 1/2$ ) if and only if  $\rho$  is said to be useful for teleportation, since it has been shown that the classical teleportation can have at most  $F = 2/3$  (or  $f = 1/2$ ) [1, 5, 9].

Let  $F_i$  be defined as the maximal teleportation fidelity on the resulting 2-qubit state in the compound system  $jk$  after the measurement of the system  $i$ , and let  $f_i$  be the maximal average of the fully entangled fraction of the state in the compound system  $jk$  after the measurement of the system  $i$ , that is,

$$f_i = \max_{U_i} \left[ \langle 0 | U_i \rho_i U_i^\dagger | 0 \rangle f(\varrho_{jk}^0) + \langle 1 | U_i \rho_i U_i^\dagger | 1 \rangle f(\varrho_{jk}^1) \right], \quad (6)$$

where the maximum is over all  $2 \times 2$  unitary matrices. Then, as in the 2-qubit case, it can be obtained [4] that for  $i \in \{1, 2, 3\}$

$$F_i = \frac{2f_i + 1}{3}. \quad (7)$$

By the reason as in the 2-qubit case, a given 3-qubit state  $\rho_{123}$  can be said to be *useful for 3-qubit teleportation* if and only if  $F_i > 2/3$  (or  $f_i > 1/2$ ) for every  $i \in \{1, 2, 3\}$ .

In order to explicitly calculate the values of  $f_i$ , we remark that a three-qubit state  $\rho_{123}$  can be described as

$$\begin{aligned} &\frac{1}{8} I \otimes I \otimes I \\ &+ \frac{1}{8} (\vec{s}_1 \cdot \vec{\sigma} \otimes I \otimes I + I \otimes \vec{s}_2 \cdot \vec{\sigma} \otimes I + I \otimes I \otimes \vec{s}_3 \cdot \vec{\sigma}) \\ &+ \frac{1}{8} \sum_{k,l=1}^3 (b_1^{kl} I \otimes \sigma_k \otimes \sigma_l + b_2^{kl} \sigma_k \otimes I \otimes \sigma_l + b_3^{kl} \sigma_k \otimes \sigma_l \otimes I) \\ &+ \frac{1}{8} \sum_{j,k,l=1}^3 t^{jkl} \sigma_j \otimes \sigma_k \otimes \sigma_l, \end{aligned} \quad (8)$$

where  $\sigma_i$  are Pauli matrices,  $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ ,  $\vec{s}_i$  are real vectors in  $\mathbb{R}^3$  satisfying  $|\vec{s}_i| \leq 1$ , and  $b_i^{kl}$  and  $t^{jkl}$  are real numbers.

For each  $i = 1, 2, 3$ , let  $\mathbf{b}_i$  be a  $3 \times 3$  real matrix with  $(k, l)$ -entry  $b_i^{kl}$ . Let  $\mathbf{T}_1^j$ ,  $\mathbf{T}_2^k$ , and  $\mathbf{T}_3^l$  be  $3 \times 3$  real matrices with  $(k, l)$ -entry  $t^{jkl}$ ,  $(j, l)$ -entry  $t^{jkl}$ , and  $(j, k)$ -entry  $t^{jkl}$ , respectively. Then by way of the results in [1], it is obtained that for each  $i = 1, 2, 3$ ,

$$f_i = \frac{1}{4} + \frac{1}{8} \max \left[ \|\mathbf{b}_i + \sum_{l=1}^3 x_l \mathbf{T}_i^l\| + \|\mathbf{b}_i - \sum_{l=1}^3 x_l \mathbf{T}_i^l\| \right], \quad (9)$$

where  $\|\cdot\| = \text{tr}|\cdot|$ , and the maximum is taken over real numbers  $x_l$  satisfying  $x_1^2 + x_2^2 + x_3^2 = 1$ . By simple calculations of the Lagrange multiplier, we have the following result: For each  $i = 1, 2, 3$ ,

$$f_i = \frac{1}{4} + \frac{1}{8} \left[ \|\mathbf{b}_i + \sum_{l=1}^3 y_l \mathbf{T}_i^l\| + \|\mathbf{b}_i - \sum_{l=1}^3 y_l \mathbf{T}_i^l\| \right], \quad (10)$$

where  $y_l = \|\mathbf{T}_i^l\| / \sqrt{\sum_t \|\mathbf{T}_i^t\|^2}$ .

For convenience, we consider the class of 3-qubit states with 4 parameters presented by Dür *et al.* [10] as follows.

$$\begin{aligned} \rho_{\text{GHZ}} &= \lambda_0^+ |\Psi_0^+\rangle \langle \Psi_0^+| + \lambda_0^- |\Psi_0^-\rangle \langle \Psi_0^-| \\ &+ \sum_{j=1}^3 \lambda_j (|\Psi_j^+\rangle \langle \Psi_j^+| + |\Psi_j^-\rangle \langle \Psi_j^-|), \end{aligned} \quad (11)$$

where  $\lambda_0^+ + \lambda_0^- + 2 \sum_j \lambda_j = 1$ , and  $|\Psi_j^\pm\rangle = (|j\rangle \pm |7-j\rangle) / \sqrt{2}$  are the GHZ-basis states. We note that any of 3-qubit states can be transformed into a state  $\rho_{\text{GHZ}}$  in the class by LOCC (the so-called depolarizing process) [10, 11]. Thus, we shall investigate teleportation capability, distillability, and nonlocality on this class of 4-parameter states, since states in the class can be representatives of 3-qubit states as Werner states [12] in 2-qubit system.

Without loss of generality, we may assume that  $\lambda_0^+$  is not less than  $\lambda_0^-$  and  $\lambda_j$ , since otherwise it can be

adjusted by a local unitary operation. Then by Eq. (10), for 4-parameter states  $\rho_{\text{GHZ}}$ , we obtain

$$\begin{aligned} f_1 &= \lambda_0^+ + \lambda_3 = 1/2 + (\lambda_0^+ - \lambda_0^-)/2 - \lambda_1 - \lambda_2, \\ f_2 &= \lambda_0^+ + \lambda_2 = 1/2 + (\lambda_0^+ - \lambda_0^-)/2 - \lambda_1 - \lambda_3, \\ f_3 &= \lambda_0^+ + \lambda_1 = 1/2 + (\lambda_0^+ - \lambda_0^-)/2 - \lambda_2 - \lambda_3. \end{aligned} \quad (12)$$

We now take the distillability over 3-qubit states into account. Note that if a 3-qubit state  $\rho$  has  $\rho^{T_j} < 0$  for all  $j = 1, 2, 3$ , where  $T_j$  represents the partial transposition for the system  $j$ , then one can distill a GHZ state from many copies of  $\rho$  by LOCC [10]. Thus, if such a state is said to be *GHZ-distillable*, then it can be obtained that a given 3-qubit state  $\rho$  is GHZ-distillable if  $N_j(\rho) > 0$  for all  $j = 1, 2, 3$ , where

$$N_j(\rho) = (\|\rho^{T_j}\| - 1)/2, \quad (13)$$

which is called the negativity, a bipartite entanglement measure [13, 14]. Then for 4-parameter states  $\rho_{\text{GHZ}}$ , we get

$$N_j = \max\{0, (\lambda_0^+ - \lambda_0^-)/2 - \lambda_{4-j}\}. \quad (14)$$

Hence, it follows from Eq. (12) and Eq. (14) that for any  $\rho_{\text{GHZ}}$ ,

$$\begin{aligned} f_1 &\leq 1/2 + N_2, \quad 1/2 + N_3, \\ f_2 &\leq 1/2 + N_1, \quad 1/2 + N_3, \\ f_3 &\leq 1/2 + N_2, \quad 1/2 + N_2. \end{aligned} \quad (15)$$

Therefore, by the inequalities in (15), we obtain the following theorem.

**Theorem 1** *If a 4-parameter state  $\rho_{\text{GHZ}}$  is useful for 3-qubit teleportation then it is GHZ-distillable.*

For the nonlocality over 3-qubit states, we consider the Mermin inequality [15] on 3-qubit states. Let  $\mathcal{B}_{\mathcal{M}}$  be the Mermin operator associated with the Mermin inequality as the following.

$$\begin{aligned} \mathcal{B}_{\mathcal{M}} &= \vec{a}_1 \cdot \vec{\sigma} \otimes \vec{a}_2 \cdot \vec{\sigma} \otimes \vec{a}_3 \cdot \vec{\sigma} - \vec{a}_1 \cdot \vec{\sigma} \otimes \vec{b}_2 \cdot \vec{\sigma} \otimes \vec{b}_3 \cdot \vec{\sigma} \\ &\quad - \vec{b}_1 \cdot \vec{\sigma} \otimes \vec{a}_2 \cdot \vec{\sigma} \otimes \vec{b}_3 \cdot \vec{\sigma} - \vec{b}_1 \cdot \vec{\sigma} \otimes \vec{b}_2 \cdot \vec{\sigma} \otimes \vec{a}_3 \cdot \vec{\sigma}, \end{aligned} \quad (16)$$

where  $\vec{a}_j$  and  $\vec{b}_j$  are unit vectors in  $\mathbb{R}^3$ . Then for a given 3-qubit state  $\rho$ , the Mermin inequality is

$$\text{tr}(\rho \mathcal{B}_{\mathcal{M}}) \leq 2. \quad (17)$$

In order that we may check whether a given state violates the Mermin inequality or not, we should calculate the maximal violation of the Mermin inequality for the state, that is, we should compute the following quantity:

$$\max_{\vec{a}_j, \vec{b}_j} \text{tr}(\rho \mathcal{B}_{\mathcal{M}}). \quad (18)$$

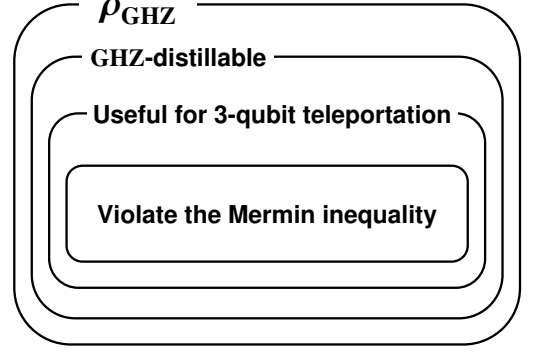


FIG. 2: The relations among the teleportation capability, distillability, and nonlocality for  $\rho_{\text{GHZ}}$ .

By direct calculations and the fact [15] that the value of Eq. (18) for the GHZ state is 4, it can be explicitly obtained that for any 4-parameter state  $\rho_{\text{GHZ}}$ ,

$$\max_{\vec{a}_j, \vec{b}_j} \text{tr}(\rho_{\text{GHZ}} \mathcal{B}_{\mathcal{M}}) = 4(\lambda_0^+ - \lambda_0^-). \quad (19)$$

Therefore, since by Eq. (19), if  $\rho_{\text{GHZ}}$  violates the Mermin inequality, then  $4(\lambda_0^+ - \lambda_0^-) > 2$ , we have the following relation between nonlocality and teleportation on  $\rho_{\text{GHZ}}$ .

**Theorem 2** *If a 4-parameter state  $\rho_{\text{GHZ}}$  violates the Mermin inequality, then  $f_i > 1/2$  for all  $i = 1, 2, 3$ , and hence it is useful for 3-qubit teleportation.*

By Theorem 1 and Theorem 2, we have the same relations among the teleportation capability, distillability, and nonlocality for  $\rho_{\text{GHZ}}$  as in 2-qubit states. The relations are seen in Fig. 2.

Since any of 3-qubit states can be transformed into a state in the class by LOCC, these relations could be also satisfied in general 3-qubit states.

In summary, we have considered the class of 3-qubit states with 4 parameters, presented by Dür *et al.* [10]. On the class, we have investigated two relations among the maximal teleportation fidelity, the distillability of the GHZ state, and the maximal violation of the Mermin inequality. We have finally shown that if any 4-parameter 3-qubit state in the class violates the Mermin inequality then the state is useful for 3-qubit teleportation, and that if any 3-qubit state in the class is useful for 3-qubit teleportation then the state can distill the GHZ state.

S.L. was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2006-003-C00044), and J.J. by the Overseas Research Student Award Program and the UK Engineering and Physical Sciences Research Council through the QIP IRC.

- 
- [1] R. Horodecki, M. Horodecki, and P. Horodecki, Phys. Lett. A **222**, 21 (1996).
- [2] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A **59**, 1829 (1999).
- [3] D.M. Greenberger, M.A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69.
- [4] S. Lee, J. Joo, and J. Kim, Phys. Rev. A **72**, 024302 (2005).
- [5] S. Popescu, Phys. Rev. Lett. **72**, 797 (1994).
- [6] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A **54**, 3824 (1996).
- [7] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).
- [8] P. Badziąg, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **62**, 012311 (2000).
- [9] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).
- [10] W. Dür, J.I. Cirac, and R. Tarrach, Phys. Rev. Lett. **83**, 3562 (1999); W. Dur and J.I. Cirac, Phys. Rev. A **61**, 042314 (2000).
- [11] Let  $\rho$  be an arbitrary 3-qubit state, and for unitary operators  $U, V, W$  in  $U(2)$ , let  $\mathcal{S}(U, V, W)$  be a superoperator defined as
- $$\mathcal{S}(U, V, W)(\rho) = \frac{1}{2}U \otimes V \otimes W \rho U^\dagger \otimes V^\dagger \otimes W^\dagger + \frac{1}{2}\rho, \quad (20)$$
- that is,  $U \otimes V \otimes W$  is performed to  $\rho$  with probability 1/2, while no operation is performed with probability 1/2. Then we first apply  $\mathcal{S}(\sigma_x, \sigma_x, \sigma_x)$  to  $\rho$ , and then apply  $\mathcal{S}(\sigma_z, \sigma_z, I)$ ,  $\mathcal{S}(\sigma_z, I, \sigma_z)$ . Then it can be easily obtained that the resulting state is diagonal in the GHZ-basis. We now apply  $\mathcal{S}(U_{3\pi/2}, U_{\pi/2}, I)$  and  $\mathcal{S}(U_{3\pi/2}, I, U_{\pi/2})$ , subsequently. Here  $U_\theta$  maps  $|j\rangle$  to  $e^{i(1-j)\theta}|j\rangle$  for  $j = 0, 1$ , when  $i = \sqrt{-1}$ . Then we can obtain the 3-qubit state with 4 parameters.
- [12] R.F. Werner, Phys. Rev. A **40**, 4277 (1989).
- [13] G. Vidal and R.F. Werner, Phys. Rev. A **65**, 032314 (2002).
- [14] S. Lee, D.P. Chi, S.D. Oh, and J. Kim, Phys. Rev. A **68**, 062304 (2003).
- [15] N.D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).